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# Conditions for the separation of the Hamilton–Jacobi equation

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**Abstract.** Necessary and sufficient conditions for the separation of the Hamilton–Jacobi equation for the geodesics in an $n$-dimensional Riemannian or pseudo-Riemannian manifold are obtained. The integrability of these conditions is investigated and several results are found which are of particular interest in the four-dimensional space–times of general relativity.

## 1. Introduction

The Hamilton–Jacobi equation for the geodesics in an $n$-dimensional Riemannian or pseudo-Riemannian manifold $V_n$ with metric tensor $g^{ij}$ is

$$g^{ij}S_{,i}S_{,j} - m^2 = 0. \tag{1.1}$$

A solution $S$ of this equation separates with respect to the coordinate $x^1$ if

$$S = S_1(x^1) + \boldsymbol{S}(x^2, x^3, \ldots, x^n). \tag{1.2}$$

The Hamilton–Jacobi equation is said to separate with respect to the coordinate $x^1$ if when (1.2) is substituted into (1.1) the resulting equation, after multiplication by an integrating factor $U$, can be written as the sum of a function of $x^1$ and a function of $x^2$, $x^3, \ldots, x^n$ for arbitrary functions $S_1$ and $\boldsymbol{S}$. A generalisation of this definition of separability is the definition used by Carter (1968) and Collinson and Fugère (1977) in obtaining empty space–times in which the Hamilton–Jacobi equation separates (further details are given in § 5). It is also the definition used by Dietz (1976). An alternative, less restrictive, definition is used by Woodhouse (1975) in his investigation of the relationship between separability and the existence of constants of motion for the geodesics. This definition is that the Hamilton–Jacobi equation is said to separate with respect to the coordinate $x^1$ if there exists a *complete* solution $S$ which separates with respect to $x^1$. In this paper the first definition will be used throughout.

In § 2 necessary and sufficient conditions for separation will be given in a covariant form. The integrability conditions for these are investigated in § 2, and the following theorem is proved.

*Theorem 1.* The Hamilton–Jacobi equation can only separate in space–times of Petrov type I, D or, provided the separable coordinate is space-like, type II.

The integrability conditions take a particularly simple form in an Einstein space and in § 3 these lead to the following theorem.

*Theorem 2.* The Hamilton–Jacobi equation can only separate in a four-dimensional Einstein space of normal hyperbolic signature if the space–time is conformally flat.

Finally, in § 5 the definition of separability is extended to spaces $V_n$ admitting an Abelian group of motions.

## 2. Covariant condition for separation

The condition (1.2) is equivalent to the differential condition

$$S_{,1\alpha}=0 \tag{2.1}$$

where $\alpha=2,3,\ldots,n$. Hence, according to the definition used here, the Hamilton–Jacobi equation will separate with respect to $x^1$ if and only if

$$(Ug^{ij}S_{,i}S_{,j}-Um^2)_{,1\alpha}=0 \tag{2.2}$$

for all functions $S$ satisfying (2.1). It is convenient to introduce the conformal space $\tilde{V}_n$ with metric

$$\tilde{g}^{ij}=Ug^{ij} \tag{2.3}$$

so that equation (2.2) becomes simply

$$(\tilde{g}^{ij}S_{,i}S_{,j}-Um^2)_{,1\alpha}=0.$$

Using (2.1) this can be written as

$$\tilde{g}^{ij}{}_{,1\alpha}S_{,i}S_{,j}+2\tilde{g}^{i\beta}{}_{,1}S_{,i}S_{,\beta\alpha}+2\tilde{g}^{il}{}_{,\alpha}S_{,i}S_{,11}+2\tilde{g}^{\beta 1}S_{,\beta\alpha}S_{,11}-m^2U_{,1\alpha}=0. \tag{2.4}$$

The necessary and sufficient conditions for the Hamilton–Jacobi equation to separate with respect to $x^1$ are that (2.4) should hold for arbitrary values of $S_{,i}$, $S_{,\beta\alpha}$ and $S_{,11}$. Hence

$$\tilde{g}^{ij}{}_{,1\alpha}=0 \tag{2.5}$$

$$\tilde{g}^{i\beta}{}_{,1}=0 \tag{2.6}$$

$$\tilde{g}^{i1}{}_{,\alpha}=0 \tag{2.7}$$

$$\tilde{g}^{\beta 1}=0 \tag{2.8}$$

and

$$U_{,1\alpha}=0. \tag{2.9}$$

The condition (2.9) implies that the integrating factor $U$ itself separates. The condition (2.5) is identically satisfied by virtue of (2.6) and (2.7) and the only component of (2.7) which is not identically satisfied by virtue of (2.8) is $\tilde{g}^{11}{}_{,\alpha}=0$. Since $\tilde{g}^{11}$ is a function of $x^1$ alone a coordinate transformation $x^1\to f(x^1)$ can be used to set $\tilde{g}^{11}=\epsilon(\pm 1)$. This transformation does not affect the separation. Thus the necessary and sufficient conditions for separation are that there exist a coordinate system such that

$$\tilde{g}^{ij}{}_{,1}=0 \tag{2.10a}$$

$$\tilde{g}^{11}=\epsilon \tag{2.10b}$$

$$\tilde{g}^{\beta 1}=0 \tag{2.10c}$$

and

$$U_{,1\alpha}=0. \tag{2.10d}$$

These conditions can be stated in a covariant form, namely that there exist a vector field $P^i$ and a scalar field $U$ satisfying

$$\underset{P}{£}\,\tilde{g}^{ij}=0 \tag{2.11a}$$

$$P^i\tilde{P}_i=\epsilon \tag{2.11b}$$

$P^i$ is hypersurface orthogonal (2.11c)

and

$$\underset{P}{£}\,U_{,i}(\delta^i_j-\epsilon P^i\tilde{P}_j)=0. \tag{2.11d}$$

Here £ is the Lie derivative, $\tilde{P}_i=\tilde{g}_{ij}P^j$ and $\delta^i_j-\epsilon P^i\tilde{P}_j$ is the projection operator which projects perpendicular to $P^i$. The equivalence of the two sets of conditions can easily be demonstrated by choosing a preferred coordinate system with $P^i=\delta^i_1$, $x^1=$ constant being the hypersurfaces orthogonal to $P^i$. If the equations (2.11) admit two commuting vectors then the coordinates can be chosen so that both $x^1$ and $x^2$ separate. This generalises to any number of commuting vectors. Written in terms of the space $V_n$ rather than the space $\tilde{V}_n$ these conditions become

$$\underset{P}{£}\,g^{ij}=-\underset{P}{£}\ln U g^{ij} \tag{2.12a}$$

$$P^iP_i=\epsilon U \tag{2.12b}$$

$P^i$ is hypersurface orthogonal (2.12c)

and

$$\underset{P}{£}\,U_{,i}(\delta^i_j-\epsilon U^{-1}P^iP_j)=0. \tag{2.12d}$$

Here $P_i=g_{ij}P^j$. It follows from equations (2.12) that $P_i$ is a conformal Killing vector for the space $V_n$ and that the separation is what Woodhouse terms orthogonal. Woodhouse shows that for such separation $P_i$ is the eigenvector of a Killing tensor. Now $U$ separates in the preferred coordinate system so that

$$U=U_1(x^1)+\boldsymbol{U}(x^\alpha).$$

The Killing tensor $T_{ij}$ associated with the separation can then be found easily, using the conformal Killing equation (2.12$a$) for $P_i$, ând can be written as

$$T_{ij}=P_iP_j+\epsilon U_1 g_{ij}. \tag{2.13}$$

Clearly $P^i$ is an eigenvector of this tensor.

## 3. The integrability conditions for separation

The conditions (2.11$a$) and (2.11$c$) can be written as

$$\tilde{P}_{i|j}+\tilde{P}_{j|i}=0$$

and

$$\tilde{P}_{i|j}-\tilde{P}_{j|i}=\phi_{,i}\tilde{P}_j-\phi_{,j}\tilde{P}_i$$

where | denotes covariant differentiation in $\tilde{V}_n$. Adding these yields

$$2\tilde{P}_{i|j} = \phi_{,i}\tilde{P}_j - \phi_{,j}\tilde{P}_i. \qquad (3.1)$$

Contracting with $P^i$ and using (2.11*b*) gives

$$\phi_{,j} = P^i\phi_{,i}\tilde{P}_j$$

so that, substituting back into (3.1),

$$\tilde{P}_{i|j} = 0. \qquad (3.2)$$

The space $\tilde{V}_n$ must therefore admit a non-null covariantly constant vector field. The integrability conditions for (3.2) is

$$\tilde{R}^h{}_{ijk}\tilde{P}_h = 0. \qquad (3.3)$$

Contracting (3.3) on $i$ and $j$ gives

$$\tilde{R}^h{}_k\tilde{P}_h = 0 \qquad (3.4)$$

and using this result it is found that (3.3) can be written in terms of the Weyl tensor $\tilde{C}^h{}_{ijk}$ as

$$\tilde{C}^h{}_{ijk}\tilde{P}_h = \frac{1}{n-2}\tilde{P}_j(\tilde{R}_{ik} - \tilde{R}\tilde{g}_{ik}/n-1) - \frac{1}{n-2}\tilde{P}_k(\tilde{R}_{ij} - \tilde{R}\tilde{g}_{ij}/n-1). \qquad (3.5)$$

Contracting equation (3.5) with $P^j$ yields

$$(\tilde{R}_{ik} - \tilde{R}\tilde{g}_{ik}/n-1) = (n-2)\epsilon\tilde{C}^h{}_{ijk}\tilde{P}_hP^j - \epsilon\tilde{P}_i\tilde{P}_k\tilde{R}/n-1. \qquad (3.6)$$

Using (3.6) to eliminate the terms involving the Ricci tensor the equation (3.5) becomes

$$\epsilon\tilde{C}^h{}_{ijk}\tilde{P}_h = \tilde{P}_k\tilde{C}^h{}_{ijs}\tilde{P}_hP^s - \tilde{P}_j\tilde{C}^h{}_{iks}\tilde{P}_hP^s. \qquad (3.7)$$

Since the Weyl tensor is conformally invariant this equation can easily be transformed into the space $V_n$ to give

$$P^sP_sC^h{}_{ijk}P_h = P_kC^h{}_{ijs}P_hP^s - P_jC^h{}_{iks}P_hP^s. \qquad (3.8)$$

The requirement that (3.8) admit a non-trivial solution for $P_i$ imposes conditions on the Weyl tensor. In a four-dimensional space–time a null tetrad can be chosen with $l_i$ a principle vector and

$$\tfrac{1}{2}U^{1/2}(l_i + \epsilon n_i) = P_i.$$

Substituting this into equation (3.8) and considering each tetrad component of the equation gives the following information about the tetrad components of the Weyl tensor:

$$\psi_0 = \psi_4 = 0, \qquad \psi_1 = -\epsilon\bar{\psi}_3 \qquad \text{and} \qquad \psi_2 = \bar{\psi}_2.$$

It follows that $l_i$ and $n_i$ are both principle vectors of the Weyl tensor and that the other principle vectors correspond to the vector $l_i'$ found by making a null rotation about $n_i$ with the parameter $a$ satisfying the quadratic equation

$$-4a^2\epsilon\bar{\psi}_3 + 6a\psi_2 + 4\psi_3 = 0. \qquad (3.9)$$

If $\psi_3 = 0$, $l_i$ and $n_i$ must be repeated principle vectors and so the Weyl tensor is of Petrov type D. If $\psi_3 \neq 0$ and the equation (3.9) has distinct roots then the principle vectors are all distinct and the Weyl tensor is of Petrov type I. Finally if $\psi_3 \neq 0$ and the equation

(3.9) has equal roots then the Weyl tensor is of Petrov type II. The condition for equal roots is that $9\psi_2^2 + 16\epsilon\psi_3\bar{\psi}_3 = 0$. Since $\psi_2$ is real this cannot hold if $\epsilon = +1$ that is if the separable coordinate is time-like. This proves theorem 1.

Written in terms of the Ricci tensor of $V_n$ the equation (3.4) becomes

$$R_{ik}P^k = (n-2)(\tfrac{1}{2}U^{-1}U_{|ij}P^j - \tfrac{3}{4}U^{-2}U_{,i}U_{,j}P^j) \\ + P_i[\tfrac{1}{2}U^{-2}\tilde{g}^{kl}U_{|kl} + \tfrac{1}{4}(n-4)U^{-3}\tilde{g}^{kl}U_{,k}U_{,l}]. \tag{3.10}$$

Equation (2.11$d$), the condition that $U$ separates, can be written as

$$U_{|ij}P^j = \epsilon P_i U^{-1}U_{|kj}P^kP^j \tag{3.11}$$

so that (3.10) can be rewritten as

$$R_{ik}P^k = -\tfrac{3}{4}(n-2)U^{-2}U_{,i}U_{,j}P^j + P_i[\tfrac{1}{2}U^{-2}\tilde{g}^{kl}U_{|kl} + \tfrac{1}{4}(n-4)U^{-3}\tilde{g}^{kl}U_{,k}U_{,l} \\ + \tfrac{1}{2}(n-2)\epsilon U^{-2}U_{|kj}P^kP^j]. \tag{3.12}$$

Contracting (3.12) with $P^i$ yields

$$\tfrac{1}{2}U^{-2}\tilde{g}^{kl}U_{|kl} + \tfrac{1}{4}(n-4)U^{-3}\tilde{g}^{kl}U_{,k}U_{,l} + \tfrac{1}{2}(n-2)\epsilon U^{-2}U_{|kj}P^kP^j \\ = \epsilon U^{-1}R_{ik}P^iP^k + \tfrac{3}{4}(n-2)U^{-3}U_{,i}P^iU_{,j}P^j. \tag{3.13}$$

Eliminating the term in square brackets in (3.12), using (3.13), gives an equation which can be rewritten as an equation for $U_{,i}$, namely

$$U_{,i} = \epsilon\alpha U^{-1}P_i + 4U^2(-R_{ij}P^j + \epsilon U^{-1}P_iR_{kl}P^kP^l)/3\alpha(n-2) \tag{3.14}$$

where

$$\alpha = U_{,j}P^j. \tag{3.15}$$

Equation (3.11) can be written in terms of $\alpha$ as

$$\alpha_{,i} = \epsilon P_i U^{-1}U_{|kj}P^kP^j. \tag{3.16}$$

An expression for $U_{|kj}P^kP^j$ can be found by differentiating (3.14) and so eliminating the term involving $\tilde{g}^{kl}U_{|kl}$ from (3.13). It follows that the derivatives of $P_i$, $U$ and $\alpha$ can all be written in terms of $P_i$ and $\alpha$ alone. Also $U = \epsilon P^iP_i$. Hence if $P_i$ and $\alpha$ are specified at a given point then $P_i$ and $U$ can be found in a neighbourhood of the point by using a Taylor series expansion. In fact $P_i$ and $\alpha$ cannot be specified arbitrarily at the given point but must satisfy certain algebraic equations. These are the so called first set of integrability conditions and are:

(i) the equation (3.8);
(ii) the integrability conditions for equations (3.14) and (3.16);
(iii) the equation obtained by writing equation (3.6) in the space $V_n$ and eliminating all derivatives of $U$.

Further integrability conditions are obtained by differentiation of the above set and eliminating derivatives of $P_i$, $U$ and $\alpha$.

## 4. The integrability conditions in an Einstein space

Consider an Einstein space, that is a space in which

$$R_{ij} = \frac{1}{n}g_{ij}R.$$

For such a space $-R_{ij}P^j+\epsilon U^{-1}P_iR_{kl}P^kP^l$ is identically zero so that the equation (3.14) simplifies considerably. In fact it is found that

$$P_{i;j}=\tfrac{1}{2}U^{-1}\alpha g_{ij} \tag{4.1}$$

$$U_{,i}=\epsilon U^{-1}\alpha P_i \tag{4.2}$$

and

$$\alpha_{,i}=\epsilon U^{-1}P_i\left(U^{-1}\alpha^2+\frac{2U^2\epsilon R}{n(n-1)}\right). \tag{4.3}$$

From these it follows that $U_{|ij}$ and $\alpha_{|ij}$ are both proportional to $P_iP_j$ so that the integrability conditions for the equations (4.2) and (4.3) are identically satisfied. Writing equation (3.6) in the space $V_n$ and eliminating all derivatives of $U$ yields the simple equation (with $n\neq 2$)

$$C^h{}_{ijk}P_hP^j=0.$$

Combining this with equation (3.8) gives

$$C^h{}_{ijk}P_h=0 \tag{4.4}$$

and this final equation (4.4) is all that remains in an Einstein space of the first set of integrability conditions. Differentiating (4.4) gives

$$C^h{}_{ijk;l}P_h+\tfrac{1}{2}\alpha U^{-1}C_{lijk}=0$$

and, provided $C_{lijk}\neq 0$, this equation will determine $\alpha$ in terms of $P_i$. Using a null tetrad (and remembering that $P_h$ is non-null) it can be shown that in a space–time equation (4.4) cannot admit a non-trivial solution for $P_h$ unless $C^h{}_{ijk}$ is zero. This proves theorem 2. Notice as a corollary that the Hamilton–Jacobi equation cannot separate in our sense in a non-flat empty space–time.

## 5. Separability in a space $V_n$ admitting an Abelian group of motions

Consider the Hamilton–Jacobi equation in three-dimensional Euclidean space with spherical polar coordinates $r$, $\theta$, $\phi$, namely

$$r^2\sin^2\theta\left(\frac{\partial S}{\partial r}\right)^2+\sin^2\theta\left(\frac{\partial S}{\partial\theta}\right)^2+\left(\frac{\partial S}{\partial\phi}\right)^2=m^2r^2\sin^2\theta. \tag{5.1}$$

This equation separates with respect to $\phi$ (with $U=1$) but not with respect to, say, $r$. Nevertheless a solution of the equation exists in which all the coordinates separate. This separation arises essentially because $\phi$ is an ignorable coordinate. Writing

$$S=c\phi+S'(r,\theta)$$

the equation (5.1) becomes

$$r^2\sin^2\theta\left(\frac{\partial S'}{\partial r}\right)^2+\sin^2\theta\left(\frac{\partial S'}{\partial\theta}\right)^2+c^2=m^2r^2\sin^2\theta. \tag{5.2}$$

This is an equation in only two variables and one can ask whether this equation separates, for arbitrary values of the constant $c$, in the sense used throughout this paper, that is can the equation, after multiplication by an integrating factor $U(r,\theta)$, be written as the sum of a function of $r$ and a function of $\theta$ whenever $S'$ is written as the sum of two such arbitrary functions. Since (5.2) is to separate for all values of the constant $c$ it follows that both the equation

$$r^2\sin^2\theta\left(\frac{\partial S'}{\partial r}\right)^2+\sin^2\theta\left(\frac{\partial S'}{\partial\theta}\right)^2=m^2r^2\sin^2\theta \tag{5.3}$$

and the integrating factor should separate. Notice that (5.3) is just the Hamilton–Jacobi equation in the subspace $\phi$ = constant and the conditions for this to separate are just the conditions found in § 2. This leads to the solution of (5.1) in which all the coordinates separate. These ideas can be used to generalise, in a space admitting an Abelian group of motions, the definition of separability given in the introduction. This generalisation is the definition used by Carter, Dietz and Collinson and Fugère.

Consider then a space $V_n$ admitting an $r$-parameter group of motions generated by the commuting Killing vectors $\underset{A}{K^i}$ $(A=n-r+1,\ldots,n)$. Suppose the coordinates are adapted to these Killing vectors so that $\underset{A}{K^i}=\delta^i_A$. Then the metric $g_{ij}$ is independent of the ignorable coordinates $x^A$. Substituting

$$S=C_Ax^A+S'(x^1,\ldots,x^{n-r})$$

into the Hamilton–Jacobi equation

$$g^{ij}S_{,i}S_{,j}-m^2=0 \tag{5.4}$$

yields an equation involving the coordinates $x^1,\ldots,x^{n-r}$ alone, namely

$$g^{IJ}S'_{,I}S'_{,J}+2g^{IA}S'_{,I}C_A+g^{AB}C_AC_B-m^2=0, \tag{5.5}$$

where $I, J=1,\ldots,n-r$. The Hamilton–Jacobi equation is said to separate with respect to the coordinates $x^1$ and $x^A$ if when the expression

$$S'=S'_1(x^1)+S'(x^2,\ldots,x^{n-r})$$

is substituted into (5.5) the resulting equation, after multiplication by an integrating factor $U(x^1,\ldots,x^{n-r})$, can be written as the sum of a function of $x^1$ and a function of $x^2, x^3,\ldots,x^{n-r}$ for arbitrary functions $S'_1$ and $S'$ and for arbitrary values of the constants $C_A$. This last condition is necessary in order to obtain a complete solution $S$ of the Hamilton–Jacobi equation. The separation of the ignorable coordinates is called trivial separation. The equation (5.5) will separate with respect to $x^1$ for all values of the constants $C_A$ if and only if the equations

$$g^{IJ}S'_{,I}S'_{,J}-m^2=0 \tag{5.6}$$

and the expressions

$$g^{IA}S'_{,I} \tag{5.7}$$

and

$$g^{AB} \tag{5.8}$$

all separate with respect to $x^1$ after multiplication by the same integrating factor $U$. Equation (5.6) is the Hamilton–Jacobi equation for the $n-r$ dimensional subspace $x^A$ = constant. A practical method of seeking separable coordinates would be to investigate the separation in this subspace using the methods of the previous sections and then to treat $g^{IA}$ and $g^{AB}$ as functions defined on the subspace and to use the separation of (5.7) and (5.8) in order to arrive at a final set of necessary and sufficient conditions for the Hamilton–Jacobi equation to separate. Alternatively the fact that (5.6), (5.7) and (5.8) should all separate implies that

$$\tilde{g}^{I\beta}{}_{,1}=0 \tag{5.9}$$

$$\tilde{g}^{I1}{}_{,\alpha}=0 \tag{5.10}$$

$$\tilde{g}^{\beta 1}=0 \tag{5.11}$$

$$\tilde{g}^{A\beta}{}_{,1}=0 \tag{5.12}$$

$$\tilde{g}^{A1}{}_{,\alpha}=0 \tag{5.13}$$

$$\tilde{g}^{AB}{}_{,1\alpha}=0 \tag{5.14}$$

and

$$U_{,1\alpha}=0. \tag{5.15}$$

From (5.10) and (5.13) the conformal metric components $\tilde{g}^{I1}$ and $\tilde{g}^{11}$ are functions of $x^1$ alone. Assuming that the separable coordinate is non-null the coordinate transformations $x^1\to f(x^1)$ and $x^A\to x^A+f^A(x^1)$ can be used to set $\tilde{g}^{11}=\epsilon(\pm 1)$ and $\tilde{g}^{I1}=0$. Then the necessary and sufficient conditions for separation are that there exists a coordinate system such that

$$\tilde{g}^{ij}{}_{,1}=\delta^i_A\delta^j_B\tilde{g}^{AB}{}_{,1} \tag{5.16a}$$

$$\tilde{g}^{11}=\epsilon \tag{5.16b}$$

$$\tilde{g}^{\beta 1}=0, \qquad \tilde{g}^{A1}=0 \tag{5.16c}$$

$$\tilde{g}^{AB}{}_{,1\alpha}=0 \tag{5.16d}$$

and

$$U_{,1\alpha}=0. \tag{5.16e}$$

These conditions can be stated in a covariant form, namely that there exists a vector field $P^i$, orthogonal to the Killing vectors which generate the Abelian group of motions, and scalar fields $U$ and $\chi^{AB}$ satisfying

$$\underset{\underset{A}{K}}{\pounds} P^i=0, \qquad \underset{\underset{A}{K}}{\pounds} U=0, \qquad \underset{\underset{A}{K}}{\pounds} \chi^{AB}=0 \tag{5.17}$$

with

$$\underset{P}{\pounds}\, \tilde{g}^{ij}=\chi^{AB}\underset{A}{K}{}^i\underset{B}{K}{}^j \tag{5.18a}$$

$$P^i\tilde{P}_i=\epsilon \tag{5.18b}$$

$P^i$ is hypersurface orthogonal (5.18$c$)

$$\chi^{AB}{}_{,i}(\delta^i_j-\epsilon P^i\tilde{P}_j)=0 \tag{5.18d}$$

and

$$\underset{P}{\pounds}\, U_{,i}(\delta^i_j-\epsilon P^i\tilde{P}_j)=0. \tag{5.18e}$$

The equivalence of the two sets of conditions can easily be demonstrated since a preferred coordinate system can be chosen with $\underset{A}{K}^i = \delta^i_A$ and $P^i = \delta^i_1$, $x^1 = \text{constant}$ being the hypersurfaces orthogonal to $P^i$. These conditions can easily be written in terms of the space $V_n$ rather than the conformal space $\tilde{V}_n$ as

$$\underset{P}{£}\, g^{ij} = -\underset{P}{£} \ln U g^{ij} + \chi^{AB} \underset{A}{K}^i \underset{B}{K}^j \tag{5.19a}$$

$$P^i P_i = \epsilon U \tag{5.19b}$$

$$P^i \text{ is hypersurface orthogonal} \tag{5.19c}$$

$$\chi^{AB}{}_{,i}(\delta^i_j - \epsilon U^{-1} P^i P_j) = 0 \tag{5.19d}$$

and

$$\underset{P}{£}\, U_{,i}(\delta^i_j - \epsilon U^{-1} P^i P_j) = 0. \tag{5.19e}$$

The separation of the coordinate $x^1$ is again orthogonal and the associated Killing tensor can be written down in the preferred coordinate system as

$$T_{ij} = P_i P_j + \epsilon U_1 g_{ij} + \psi^{AB} \underset{A}{K}_i \underset{B}{K}_j \tag{5.20}$$

where

$$\psi^{AB} = -\int \chi^{AB}(x^1)\, \mathrm{d}x^1.$$

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